Q: A particle of kinetic energy E impinges on a potential step of height V where $E > V$. Determine the reflection coefficient R and the transmission coefficient T.

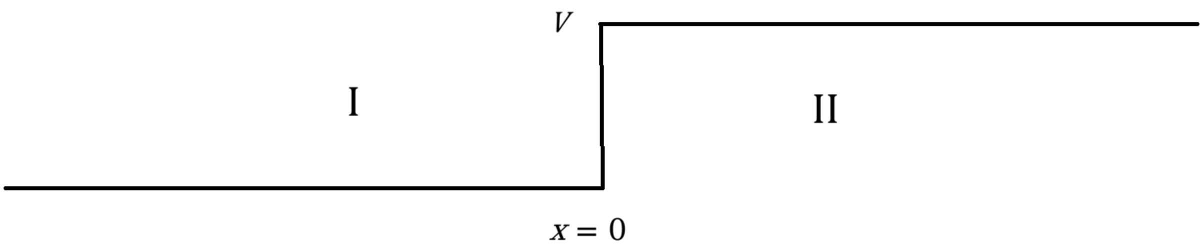
A: A free particle traveling to the right can be written as a plane wave

$$\Psi(x,t) = Ae^{i(k_1 x - \omega t)}$$

where the wavenumber

$$k_1 = \frac{\sqrt{2mE}}{\hbar}$$

Let us assume that the step potential is located at at $x = 0$ and that the particle reaches it at $t = 0$.



In region I there is also a reflected (leftward) wave

$$\Psi(x,t) = Be^{i(-k_1 x - \omega t)}$$

and in region II the transmitted wave is

$$\Psi(x,t) = Ce^{i(k_2 x - \omega t)}$$

with

$$k_2 = \frac{\sqrt{2m(E-V)}}{\hbar}$$

At $t = 0$ the sum of the wavefunctions and its dervative with respect to x must be continuous at the boundary between region I and region II, so

$$A + B = C$$

$$Aik_1 - Bik_1 = Cik_2$$

Solving these equations simultaneously yields

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2} \quad \text{and} \quad \frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

These are the ratios of the wave amplitudes. The reflection and transmission coefficients are defined as the ratio of the reflected and transmitted probability *current*—which depends not only on the amplitudes but also their momenta.

$$R = \left| \frac{J_{\text{reflected}}}{J_{\text{incident}}} \right| \quad \text{and} \quad T = \left| \frac{J_{\text{transmitted}}}{J_{\text{incident}}} \right|$$

Using the definition

$$J = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right)$$

it can be shown that

$$J_{\text{incident}} = \frac{\hbar k_1}{m} |A|^2$$

$$J_{\text{reflected}} = \frac{\hbar k_1}{m} |B|^2$$

$$J_{\text{transmitted}} = \frac{\hbar k_2}{m} |C|^2$$

And therefore we have

$$R = \left(\frac{B}{A} \right)^2 = \boxed{\left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2}$$

and

$$T = \frac{k_2}{k_1} \left(\frac{C}{A} \right)^2 = \boxed{\frac{4k_1 k_2}{(k_1 + k_2)^2}}$$

The factor $\frac{k_2}{k_1}$ can be thought of the ratio of the impedences of the two regions.